

COMPANION MATRICES AND THEIR RELATIONS TO TOEPLITZ AND HANKEL MATRICES A

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Abstract. In this paper we describe some properties of companion matrices and demonstrate some special patterns that arise when a Toeplitz or a Hankel matrix is multiplied by a related companion matrix. We present a new condition, generalizing known results, for a Toeplitz or a Hankel matrix to be the transforming matrix for a similarity between a pair of companion matrices. A special case of our main result shows that a Toeplitz or a Hankel matrix can be extended using associated companion matrices, preserving the Toeplitz or Hankel structure respectively.

Key words. Companion matrix, Toeplitz matrix, Hankel matrix, Bezoutian

AMS subject classifications. 15A21, 15A24, 15A99

1. Introduction and notation. Companion matrices occur in many scientific fields. In particular, a companion matrix naturally arises as the system matrix when a dynamic system is represented in state space form [6, 9]. When a basis of the state vector space is changed a new system matrix appears and, very often, the new system matrix is also a companion matrix and the similarity relation between the new and the old system matrices is realized by a nonsingular Toeplitz (or Hankel) matrix. In the literature on dynamic systems such similarity transformations are verified case by case. In this paper we give a general condition for a Toeplitz or Hankel matrix, satisfaction of which ensures that the Toeplitz (Hankel) matrix transforms one companion matrix to another. In the second part of this paper we also investigate some extensions of a matrix by companion matrices, and finally through an example we indicate some applications. In a dynamic system, if the system matrix under a basis of an n -dimensional state space is a companion matrix, then a special case of the extension we introduce yields the well known representation of the state vector at any future time instant in terms of a given initial state vector together with knowledge of the input up to the current time. We discuss applications in a discrete-time setting. Continuous-time analogues are easily deduced, and are virtually identical.

Given vectors $\mathbf{u} = (u_1, \dots, u_{n+1})^T$ and $\mathbf{v} = (v_1, \dots, v_{n+1})^T \in \mathbb{R}^{n+1}$ we define the polynomials

$$u(\lambda) = u_1 + u_2\lambda + \dots + u_n\lambda^{n-1} + u_{n+1}\lambda^n$$

and

$$v(\lambda) = v_1 + v_2\lambda + \dots + v_n\lambda^{n-1} + v_{n+1}\lambda^n.$$

We assume always that u_1 , u_{n+1} , v_1 and v_{n+1} are nonzero, and that $u(\lambda)$ and $v(\lambda)$ are co-prime. The “top”, “bottom”, “left” and “right” companion matrices of the polynomial $u(\lambda)$ (or the vector \mathbf{u}) are defined as

$$C_t(\mathbf{u}) := \begin{bmatrix} -\frac{u_n}{u_{n+1}} & \dots & -\frac{u_1}{u_{n+1}} \\ I_{n-1} & & 0 \end{bmatrix}, \quad C_b(\mathbf{u}) := \begin{bmatrix} 0 & I_{n-1} \\ -\frac{u_{n+1}}{u_1} & \dots & -\frac{u_2}{u_1} \end{bmatrix},$$

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$$C_l(\mathbf{u}) := \begin{bmatrix} -\frac{u_2}{u_1} & & & \\ & I_{n-1} & & \\ \vdots & & & \\ -\frac{u_{n+1}}{u_1} & & 0 & \end{bmatrix} \quad \text{and} \quad C_r(\mathbf{u}) := \begin{bmatrix} 0 & -\frac{u_1}{u_{n+1}} & & \\ & \vdots & & \\ I_{n-1} & -\frac{u_n}{u_{n+1}} & & \end{bmatrix}.$$

When their dependence on \mathbf{u} is clear from context we will simply write C_t , C_b , C_l and C_r . The companion matrices of $v(\lambda)$ are defined in the same way. Under our assumptions on \mathbf{u} and \mathbf{v} , all the companion matrices defined above are nonsingular.

Let J be the flipping matrix

$$J = \begin{bmatrix} 0 & & 1 \\ & \ddots & \\ 1 & & 0 \end{bmatrix}.$$

For a vector \mathbf{u} we denote by \mathbf{u}^J the vector $J\mathbf{u}$, and corresponding polynomial $u^J(\lambda)$ is defined by $u^J(\lambda) = u_{n+1} + u_n\lambda + \cdots + u_2\lambda^{n-1} + u_1\lambda^n$. For a matrix A we denote by A^J the flipping of A about its secondary diagonal, so $A^J = JA^TJ$. Hankel matrices are symmetric in the usual sense but Toeplitz matrices A are persymmetric, that is, symmetric about their secondary diagonal

$$(1.1) \quad A^J = A.$$

We also define the companion matrices of \mathbf{u}^J and denote them by $C_t(\mathbf{u}^J)$, $C_b(\mathbf{u}^J)$, $C_l(\mathbf{u}^J)$ and $C_r(\mathbf{u}^J)$. When their dependence on \mathbf{u}^J is clear from context we will simply write these matrices as \overline{C}_t , \overline{C}_b , \overline{C}_l and \overline{C}_r .

Define the following triangular Toeplitz matrices using the components of \mathbf{u} and \mathbf{v} :

$$U_+ := \begin{bmatrix} u_1 & 0 & \cdots & 0 \\ u_2 & u_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_n & \cdots & u_2 & u_1 \end{bmatrix} \quad U_- := \begin{bmatrix} u_{n+1} & u_n & \cdots & u_2 \\ 0 & u_{n+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & u_n \\ 0 & \cdots & 0 & u_{n+1} \end{bmatrix}.$$

Analogously we define V_+ and V_- in terms of the components of \mathbf{v} .

The Toeplitz Bezoutian $B_T := \mathbf{Bez}_T(\mathbf{u}, \mathbf{v}) = (b_{ij})_{i,j=1}^n$ and Hankel Bezoutian $B_H := \mathbf{Bez}_H(\mathbf{u}, \mathbf{v}) = (b_{ij})_{i,j=1}^n$ of the vectors \mathbf{u}, \mathbf{v} (or the polynomials $u(\lambda), v(\lambda)$) are the $n \times n$ matrices with the generating polynomials

$$(1.2) \quad \sum_{i,j=1}^n b_{ij} \lambda^{i-1} \mu^{j-1} = \frac{u(\lambda) v^J(\mu) - u^J(\mu) v(\lambda)}{1 - \mu\lambda}$$

and

$$(1.3) \quad \sum_{i,j=1}^n b_{ij} \lambda^{i-1} \mu^{j-1} = \frac{u(\lambda) v(\mu) - u(\mu) v(\lambda)}{\lambda - \mu}$$

respectively. The Gohberg-Semencul formulae [3, 4] imply that the Toeplitz Bezoutian matrix generated by \mathbf{u} and \mathbf{v} is

$$(1.4) \quad B_T = U_+ V_- - V_+ U_- = V_- U_+ - U_- V_+,$$

and the Hankel Bezoutian matrix generated by \mathbf{u} and \mathbf{v} is

$$(1.5) \quad B_H = V_+ J U_- - U_+ J V_- = U_- J V_+ - V_- J U_+.$$

It is known [7] that if $u(\lambda)$ and $v(\lambda)$ are co-prime then B_T and B_H are both nonsingular and that B_T^{-1} is Toeplitz and B_H^{-1} is Hankel.

2. Properties of companion matrices. Here we list some obvious relations among the companion matrices defined in Section 1.

Properties:

1. Inversion:

$$(2.1) \quad C_t = C_b^{-1}, \quad C_l = C_r^{-1}, \quad \overline{C}_t = \overline{C}_b^{-1}, \quad \overline{C}_l = \overline{C}_r^{-1}.$$

2. Flipping:

$$(2.2) \quad C_t^J = C_r, \quad C_b^J = C_l, \quad \overline{C}_t^J = \overline{C}_r, \quad \overline{C}_b^J = \overline{C}_l.$$

3. Transposition:

$$(2.3) \quad C_t^T = \overline{C}_l, \quad C_b^T = \overline{C}_r, \quad C_r^T = \overline{C}_b, \quad C_l^T = \overline{C}_t.$$

Property 1 can be found in [2]. All others can be easily verified.

2.1. Similarity. There are many similarities among the companion matrices. We are interested here in finding matrices which realize such similarities. An obvious one (see [6]), easily verifiable by a simple calculation, is U_+ , because

$$U_+ C_t U_+^{-1} = C_r \quad \text{and} \quad U_+ C_b U_+^{-1} = C_l.$$

Since $C_b = C_t^{-1}$ and $C_l = C_r^{-1}$, the second equation can be written as $U_+ C_t^{-1} U_+^{-1} = C_r^{-1}$. If, for any positive integer k we write $C_t^{-k} = (C_t^{-1})^k = C_b^k$ and $C_r^{-k} = (C_r^{-1})^k = C_l^k$, then the similarity relation above extends to:

$$(2.4) \quad U_+ C_t^k U_+^{-1} = C_r^k$$

for all integers k .

Another nontrivial similarity relation, this time between C_t^T and C_t , is also given in [6], where a linear dynamic system in continuous time is represented in the state space form

$$\dot{\alpha}(t) = A\alpha + wB, \quad y = D\alpha,$$

where $\alpha(t)$ is the state vector, A is the system matrix, B is the input column matrix, D is the output row matrix, w is a scalar input and y is the scalar output. The state space representation is said to be in canonical observer form if the system matrix A_1 , the input matrix B_1 and the output matrix D_1 are

$$A_1 = \begin{bmatrix} -a_1 & \cdots & -a_n \\ & I_{n-1} & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad D_1 = [b_1 \quad \cdots \quad b_n].$$

In our notation $A_1 = C_t^T(\mathbf{u})$ with $\mathbf{u} = (a_n, \dots, a_1, 1)^T$. The system representation is in canonical controller form if $A_2 = C_t(\mathbf{u})$, $B_2 = (b_1, \dots, b_n)^T$ and the output matrix is $D_1 = (1, 0, \dots, 0)$. In the canonical observer form the controllability matrix $\mathcal{C}(A_1, B_1)$ and the observability matrix $\mathcal{O}(D_1, A_1)$ are then constructed as follows:

$$\mathcal{C}(A_1, B_1) = [B_1 \quad A_1 B_1 \quad \dots \quad A_1^{n-1} B_1]$$

and

$$\mathcal{O}(D_1, A_1) = \begin{bmatrix} D_1 \\ D_1 A_1 \\ \vdots \\ D_1 A_1^{n-1} \end{bmatrix}.$$

In the canonical controller form $\mathcal{C}(A_2, B_2)$ and $\mathcal{O}(D_2, A_2)$ can be constructed in the same way. Under the condition that the system is both controllable and observable it can be shown that the matrix

$$Q = \mathcal{O}^{-1}(D_2, A_2) \mathcal{O}(D_1, A_1) = \mathcal{C}(A_2, B_2) \mathcal{C}^{-1}(A_1, B_1)$$

will transform A_2 into A_1 by way of $Q^{-1} A_2 Q = A_1$, that is

$$Q^{-1} C_t Q = C_t^T = \overline{C}_l.$$

It is shown in [6] that $Q = -B_T^T J$ where B_T is the Toeplitz Bezoutian $\mathbf{Bez}_T(\mathbf{u}, \mathbf{v})$ and hence Q is a Hankel Bezoutian. To make our notation consistent with the notation in [6] we have put $\mathbf{u} = (a_n, \dots, a_1, 1)^T$ and $\mathbf{v} = (b_n, \dots, b_1, 0)^T$. In general, for all integers k

$$(2.5) \quad Q C_t^k Q^{-1} = \overline{C}_l^k.$$

In this section we introduce a general condition for a nonsingular matrix to be a transforming matrix realizing a similarity between companion matrices. We will show that both (2.4) and (2.5) are special cases of our general result.

To describe our generalization, we define a simple operation on square Toeplitz or Hankel matrices. For an invertible Toeplitz matrix

$$T = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{1-n} \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{-1} \\ a_{n-1} & \cdots & a_1 & a_0 \end{bmatrix},$$

the $(n-1) \times (n+1)$ Toeplitz matrix ∂T , introduced by [5], is obtained by adding one column to the right preserving the Toeplitz structure and then deleting the first row:

$$(2.6) \quad \partial T := \begin{bmatrix} a_1 & a_0 & a_{-1} & \cdots & a_{1-n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{n-1} & \cdots & a_1 & a_0 & a_{-1} \end{bmatrix}.$$

Similarly, for an invertible Hankel matrix $H = T J$, the $(n-1) \times (n+1)$ Hankel matrix ∂H is obtained by adding one column to the right preserving the Hankel structure

and then deleting the last row:

$$(2.7) \quad H = \begin{bmatrix} a_{1-n} & \cdots & a_{-1} & a_0 \\ \vdots & \ddots & a_0 & a_1 \\ a_{-1} & \ddots & \ddots & \vdots \\ a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}, \quad \partial H := \begin{bmatrix} a_{1-n} & \cdots & a_{-1} & a_0 & a_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{-1} & a_0 & a_1 & \cdots & a_{n-1} \end{bmatrix}.$$

THEOREM 2.1. *Suppose T is an invertible $n \times n$ Toeplitz matrix, and $\mathbf{u} = (u_1, \dots, u_{n+1})^T$ is a vector such that $u_1, u_{n+1} \neq 0$. Then the following three statements are equivalent.*

1. $\mathbf{u} \in \text{Ker}\{\partial T\}$.
2. Both $C_t T$ and $T C_r$ are Toeplitz matrices and satisfy $C_t T = T C_r$.
3. Both $C_b T$ and $T C_l$ are Toeplitz matrices and satisfy $C_b T = T C_l$.

Furthermore, for all integers k ,

$$(2.8) \quad T^{-1} C_t^k T = C_r^k.$$

Proof. We prove that item 1 implies item 2 first. We use the notation $T_{[i:j, k:l]}$ to denote the sub-matrix of T formed by selecting all rows from the i th row to the j th row and all columns from the k th column to the l th column. It is easy to see that

$$T C_r = \begin{bmatrix} T_{[1:n, 2:n]} & \beta \end{bmatrix}$$

where β is a column given by

$$\beta = -\frac{1}{u_{n+1}} T \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = -\frac{1}{u_{n+1}} \begin{bmatrix} T_{[1:1, 1:n]} \\ T_{[2:n, 1:n]} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.$$

Since $\mathbf{u} \in \text{Ker}\{\partial T\}$ we have

$$T_{[2:n, 1:n]} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + u_{n+1} \begin{bmatrix} a_{1-n} \\ \vdots \\ a_{-1} \end{bmatrix} = 0.$$

This implies that

$$\beta = \begin{bmatrix} \mu_{-1} \\ a_{1-n} \\ \vdots \\ a_{-1} \end{bmatrix}$$

where $\mu_{-1} = -\frac{1}{u_{n+1}} \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{1-n} & 0 \end{bmatrix} \mathbf{u}$, that is, $\beta^T J$ is the first row of $T C_r$. From this we conclude that $T C_r$ is Toeplitz and hence $T C_r = (T C_r)^J$. On the other hand, since T is Toeplitz, by equation (2.2) we have

$$(T C_r)^J = J (T C_r)^T J = (J (C_r)^T J) (J T^T J) = (C_r)^J T^J = C_t T.$$

As a consequence we have

$$T C_r = C_t T.$$

Next we prove that item 2 implies item 1. By equating the last columns of TC_r and $C_t T$ we see that

$$-\frac{1}{u_{n+1}}T \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} \mu_1 \\ a_{1-n} \\ \vdots \\ a_{-1} \end{bmatrix}.$$

Deleting the first row on both sides we obtain $(\partial T)\mathbf{u} = 0$.

Finally we prove that item 3 is equivalent to item 1. Similar to the argument above we have

$$TC_l = \begin{bmatrix} \gamma & T_{[1:n, 1:n-1]} \end{bmatrix}$$

where γ is a column given by

$$\gamma = -\frac{1}{u_1}T \begin{bmatrix} u_2 \\ \vdots \\ u_{n+1} \end{bmatrix} = -\frac{1}{u_1} \begin{bmatrix} T_{[1:n-1, 1:n]} \\ T_{[n-1:n, 1:n]} \end{bmatrix} \begin{bmatrix} u_2 \\ \vdots \\ u_{n+1} \end{bmatrix}.$$

Since $\mathbf{u} \in \text{Ker}\{\partial T\}$ we have

$$u_1 \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} + T_{[1:n-1, 1:n]} \begin{bmatrix} u_2 \\ \vdots \\ u_{n+1} \end{bmatrix} = 0.$$

This implies that

$$\gamma = \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \\ \mu_1 \end{bmatrix}$$

where $\mu_1 = -\frac{1}{u_1} \begin{bmatrix} 0 & a_n & a_{n-1} & \cdots & a_1 \end{bmatrix} \mathbf{u}$, that is, $\gamma^T J$ is the last row of TC_l . From this we conclude that TC_l is Toeplitz and hence $TC_l = (TC_l)^J$. On the other hand, since T is Toeplitz, by equation (2.2) we have

$$(TC_l)^J = J(TC_l)^T J = (J(C_l)^T J)(JT^T J) = (C_l)^J T^J = C_b T.$$

As a consequence we have

$$TC_l = C_b T.$$

Thus item 1 implies item 3. To see that item 3 implies item 1 we equate the first columns of TC_l and $C_b T$ to get

$$-\frac{1}{u_1}T \begin{bmatrix} u_2 \\ \vdots \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} a_1 \\ \vdots \\ a_{n-1} \\ \mu_1 \end{bmatrix}.$$

Deleting the last row on both sides we obtain $(\partial T)\mathbf{u} = 0$.

To see (2.8) we write $TC_r = C_t T$ in the form $T^{-1}C_t T = C_r$ and then take integer powers on both sides.

□

COROLLARY 2.2. *Suppose H is an invertible $n \times n$ Hankel matrix and $\mathbf{u} = (u_1, \dots, u_{n+1})^T$ is a vector such that $u_1, u_{n+1} \neq 0$. Then the following statements are equivalent.*

1. $\mathbf{u}^J \in \text{Ker}\{\partial H\}$.
2. Both $C_t H$ and $H \overline{C}_l$ are Hankel matrices and satisfy $C_t H = H \overline{C}_l$.
3. Both $C_b H$ and $H \overline{C}_r$ are Hankel matrices and satisfy $C_b H = H \overline{C}_r$.

Furthermore, for all integers k ,

$$(2.9) \quad H^{-1} C_b^k H = \overline{C}_l^k.$$

Proof. Let $T = HJ$. Then T is a non-singular Toeplitz matrix. Obviously $\mathbf{u}^J \in \text{Ker}\{\partial H\}$ is equivalent to $\mathbf{u} \in \text{Ker}\{\partial T\}$. Now we prove that item 2 of Theorem 2.1 and item 2 of Corollary 2.2 are equivalent, that is, $TC_r = C_t T$ is equivalent to $C_t H = H \overline{C}_l$. Since H is Hankel we then have

$$T^J = JT^T J = (TJ)^T J = H^T J = HJ.$$

By taking the flipping operation on both sides of $TC_r = C_t T$ we obtain $C_r^J T^J = T^J C_t^J$ which is, by (2.2), $C_t H = H C_t^T$. Applying (2.3) we have

$$C_t H = H \overline{C}_l.$$

The proof for the equivalence of item 3 of Theorem 2.1 and Corollary 2.2 follows in a similar way. This completes the proof. □

COROLLARY 2.3. *If A is an invertible matrix such that $A^{-1}C_t A = C_r$, then $B = C_t A$ (or $B = AC_r$) plays the same role as A , that is, $B^{-1}C_t B = C_r$.*

If A is an invertible matrix such that $A^{-1}C_t A = \overline{C}_l$, then $B = C_t A$ (or $B = A \overline{C}_r$) plays the same role as A , that is, $B^{-1}C_t B = \overline{C}_l$.

The proof of Theorem 2.1 is constructive. Given a pair of companion matrices as stated in the Theorem, we can use the procedure in the proof to find a Toeplitz (or Hankel) matrix in our preferred pattern to perform the similarity transformation. For example, if we want to find a lower triangular Toeplitz matrix T such as $T^{-1}C_t(\mathbf{u})T = C_r(\mathbf{u})$, it turns out that T is actually a scalar multiple of U_+ . This also confirms that (2.4) is a special case of Theorem 2.1. We demonstrate this in the following example instead of giving a general proof. The general proof follows from the same argument.

We will now illustrate this with an example. Let $\mathbf{u} = [4, 3, 2, 1]^T$. Then

$$C_t(\mathbf{u}) := \begin{bmatrix} -2 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad C_r(\mathbf{u}) := \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix}.$$

Suppose we wish to construct a lower triangular Toeplitz T such that

$$T^{-1}C_t(\mathbf{u})T = C_r(\mathbf{u}).$$

First we find a Toeplitz T_1 by using the procedure given in the proof. All we need to do is to find a ∂T_1 and then obtain T_1 from ∂T_1 . ∂T_1 is a 2×4 Toeplitz matrix

whose rows are both perpendicular to \mathbf{u} . For convenience, we choose the first row to be $\begin{bmatrix} 1/4 & 0 & 0 & -1 \end{bmatrix}$. Then ∂T_1 takes the form

$$\begin{bmatrix} 1/4 & 0 & 0 & -1 \\ x & 1/4 & 0 & 0 \end{bmatrix}$$

where x is determined so that the second row is also perpendicular to \mathbf{u} , and hence $x = -3/16$. Therefore

$$T_1 = \begin{bmatrix} 0 & 0 & -1 \\ 1/4 & 0 & 0 \\ -3/16 & 1/4 & 0 \end{bmatrix}.$$

To obtain a lower triangular Toeplitz matrix we apply Corollary 2.3 to T_1 :

$$T = T_1 C_l = \begin{bmatrix} 1/4 & 0 & 0 \\ -3/16 & 1/4 & 0 \\ 1/64 & -3/16 & 1/4 \end{bmatrix}.$$

We can check that $T = U_+^{-1}$ and hence the similarity (2.4) holds.

Now we turn to the special case (2.5) of Theorem 2.1, where the similarity is carried out by the Toeplitz Bezoutian. For this purpose we first collect some results from Corollary 2.3, 2.10, Theorem 4.2 and 4.5 of [4] and summarize them in the following Theorems.

THEOREM 2.4. *A necessary and sufficient condition for two non-zero Toeplitz Bezoutian matrices $B_T(\mathbf{a}, \mathbf{b})$ and $B_T(\mathbf{a}_1, \mathbf{b}_1)$ to coincide is*

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{b}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} \varphi$$

for some matrix φ with $\det \varphi = 1$.

If T is an invertible Toeplitz matrix and $\{\mathbf{a}, \mathbf{b}\}$ is a basis for the kernel of ∂T . Then $B_T(\mathbf{a}, \mathbf{b})$ is just a scalar multiple of T^{-1} .

THEOREM 2.5. *A necessary and sufficient condition for two non-zero Hankel Bezoutian matrices $B_H(\mathbf{a}, \mathbf{b})$ and $B_H(\mathbf{a}_1, \mathbf{b}_1)$ to coincide is*

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{b}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} \varphi$$

for some matrix φ with $\det \varphi = 1$.

If H is an invertible Hankel matrix and $\{\mathbf{a}, \mathbf{b}\}$ is a basis for the kernel of ∂H . Then $B_H(\mathbf{a}, \mathbf{b})$ is just a scalar multiple of H^{-1} .

Putting $T = B_T^{-1}$ in Theorem 2.4 we see that if $\{\mathbf{a}, \mathbf{b}\}$ is a basis for the kernel of $\partial(B_T^{-1})$ then $B_T(\mathbf{a}, \mathbf{b}) = \lambda B_T(\mathbf{u}, \mathbf{v})$ for some nonzero constant λ . By Theorem 2.4 again we have

$$\begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix} \varphi$$

for some invertible matrix φ . This means that both \mathbf{u} and \mathbf{v} are in the kernel of ∂T and hence Theorem 2.1 implies

$$(2.10) \quad B_T C_t(\mathbf{u})^k B_T^{-1} = C_r(\mathbf{u})^k, \quad B_T C_t(\mathbf{v})^k B_T^{-1} = C_r(\mathbf{v})^k$$

for all integers k . Now we can show that relation (2.5) is nothing but the first equation in (2.10). To see this we rewrite the first equation in the form $C_t = B_T^{-1} C_r B_T$ and then take transpose

$$\overline{C}_t = C_t^T = C_r^J = B_T^T C_r^T (B_T^{-1})^T = (B_T^T J)(C_r)^J (B_T^T J)^{-1} = Q C_t Q^{-1}$$

which is (2.5).

2.2. Extension using companion matrices. A Toeplitz (or Hankel) matrix can be extended to any size in a Toeplitz (or Hankel) way, by adding more diagonal bands to the existing bands. Theoretical aspects of such an extension, such as the minimum rank of the extension, have been studied in the literature (see [1] and the references therein). What we are concerned with here is a specific way of extending the matrix by multiplication by some associated companion matrices. We hope this extension might have more applications than the ones we will demonstrate at the end of this paper. We will use the similarity relations among companion matrices that have been developed earlier.

Assume A is a $n \times n$ matrix. The role that C_t plays in the product $C_t A$ is to keep the first $n - 1$ rows of A as the last $n - 1$ rows of $C_t A$, and to add one new row on the top. The new row added is a linear combination of rows of A . Similarly, the first $n - 1$ rows of $C_b A$ are the last $n - 1$ rows of A and the last row of $C_b A$ is a linear combination of rows of A . This enables us to extend the matrix A in the upward and downward directions as follows. Starting from A , for integers $k \geq l$ we define $\mathcal{T}[A : k, l]$ to be the $(n + k - l) \times n$ matrix

$$(2.11) \quad \mathcal{T}[A : k, l] = \begin{bmatrix} \gamma_{k-l} \\ \vdots \\ \gamma_1 \\ C_t^l A \end{bmatrix}$$

where γ_i ($i = 1, 2, \dots, k - l$) is the first row of $C_t^{l+i} A$.

In similar fashion the effect of post-multiplying a matrix by C_r or C_l can be considered. We can extend a matrix in the right and left directions by adding the last column of AC_r to the right or adding the first column of AC_l to the left. Starting from $\mathcal{T}[A : k, l]$, for integers $s \geq t$ we define

$$(2.12) \quad \mathcal{T}[A : k, l; s, t] = \begin{bmatrix} \mathcal{T}[A : k, l] C_r^t & \beta_1 & \cdots & \beta_{s-t} \end{bmatrix},$$

where β_i ($i = 1, 2, \dots, s - t$) is the last column of $\mathcal{T}[A : k, l] C_r^{t+i}$. We call this a Toeplitz extension because we will prove that, under certain conditions, such an extension preserves the Toeplitz structure if the starting matrix A is Toeplitz. We call A a generator in such an extension. To generate the same matrix $\mathcal{T}[A : k, l; s, t]$ we can use any $n \times n$ matrix of the form $C_t^i A C_r^j$ where i and j are integers. It is easy to see that

$$\mathcal{T}[A : k, l; s, t] = \mathcal{T}[C_t^i A C_r^j : k - i, l - i; s - j, t - j].$$

If we use $\overline{C_r}$ and $\overline{C_l}$ instead of C_r and C_l in the above extension, we will obtain a different extended matrix $\mathcal{H}[A : k, l; s, t]$. We call this a Hankel extension because it preserves the Hankel structure under certain conditions.

Here are two examples:

$$\mathcal{T}[I : n, -n; n, -n] = \begin{bmatrix} C_t^n C_l^n & C_t^n & C_t^n C_r^n \\ C_l^n & I & C_r^n \\ C_b^n C_l^n & C_b^n & C_b^n C_r^n \end{bmatrix},$$

$$\mathcal{H}[A : n, -n; -n, -2n] = \begin{bmatrix} C_t^n \overline{A C_r}^{-2n} & C_t^n \overline{A C_r}^{-n} \\ \overline{A C_r}^{-2n} & \overline{A C_r}^{-n} \\ C_b^n \overline{A C_r}^{-2n} & C_b^n \overline{A C_r}^{-n} \end{bmatrix} = \begin{bmatrix} C_t^n \overline{A C_l}^{2n} & C_t^n \overline{A C_l}^n \\ \overline{A C_l}^{2n} & \overline{A C_l}^n \\ C_b^n \overline{A C_l}^{2n} & C_b^n \overline{A C_l}^n \end{bmatrix}.$$

We notice that, for any square matrix A ,

$$\mathcal{T}[A : k, l; s, t] = \mathcal{T}[I : k, l; 0, 0]A\mathcal{T}[I : 0, 0; s, t].$$

An obvious property of these extensions is given in the following Proposition.

PROPOSITION 2.6. *Suppose A is invertible. Then the rank of $\mathcal{T}[A : k, l; s, t]$ is n . If $r = s - t > 0$ then $\{e_1, \dots, e_r\}$ is a basis for the kernel of $\mathcal{T}[A : k, l; s, t]$, where e_i is the i th column of the $(n + r) \times r$ Toeplitz matrix whose first column is $\begin{bmatrix} u_1 & \cdots & u_{n+1} & 0 & \cdots & 0 \end{bmatrix}^T$ and last column is $\begin{bmatrix} 0 & \cdots & 0 & u_1 & \cdots & u_{n+1} \end{bmatrix}^T$. In particular, $\mathcal{T}[A : k, k; s, s - 1]\mathbf{u} = 0$ for all integers k and s .*

Proof. All the rows of $\mathcal{T}[A : k, l; s, t]$ are linear combinations of the rows of $\mathcal{T}[A : 0, 0; s, t]$ which is a rank n matrix. Thus $\mathcal{T}[A : k, l; s, t]$ is of rank n . It also follows that the kernel of $\mathcal{T}[A : k, l; s, t]$ is the same as the kernel of $\mathcal{T}[A : 0, 0; s, t]$. The latter is an $n \times (n + r)$ matrix so its kernel has dimension r . It is clear that the set $\{e_1, \dots, e_r\}$ is linearly independent. So the only thing we need to verify is $\mathcal{T}[A : 0, 0; s, t]e_i = 0$ for $i = 1, \dots, r$. Due to the structure of e_i

$$\mathcal{T}[A : 0, 0; s, t]e_i = \begin{bmatrix} AC_r^{s+i-1} & \mathbf{a} \end{bmatrix} \mathbf{u}$$

where \mathbf{a} is the last column of $AC_r^{s+i} = AC_r^{s+i-1}C_r$. Therefore $\mathbf{a} = AC_r^{s+i-1}\mathbf{b}$ where \mathbf{b} is the last column of C_r . A direct verification yields $\begin{bmatrix} I & \mathbf{b} \end{bmatrix} \mathbf{u} = 0$. As a consequence

$$\begin{bmatrix} AC_r^{s+i-1} & \mathbf{a} \end{bmatrix} \mathbf{u} = \begin{bmatrix} AC_r^{s+i-1} & AC_r^{s+i-1}\mathbf{b} \end{bmatrix} \mathbf{u} = AC_r^{s+i-1} \begin{bmatrix} I & \mathbf{b} \end{bmatrix} \mathbf{u} = 0.$$

□

COROLLARY 2.7. *Suppose A is invertible. Then the rank of $\mathcal{H}[A : k, l; s, t]$ is n . If $r = s - t > 0$ then $\{e_1^J, \dots, e_r^J\}$ is a basis for the kernel of $\mathcal{H}[A : k, l; s, t]$, where e_i^J is the i th column of the $(n + r) \times r$ Hankel matrix whose first column is $\begin{bmatrix} 0 & \cdots & 0 & u_{n+1} & \cdots & u_1 \end{bmatrix}^T$ and last column is $\begin{bmatrix} u_{n+1} & \cdots & u_1 & 0 & \cdots & 0 \end{bmatrix}^T$.*

Proof. Use \overline{C}_r and e_i^J instead of C_r and e_i in the proof of Proposition 2.6. □

The following Lemma is a preparation for the proof of our main Theorem 2.10.

LEMMA 2.8. *Let T be a Toeplitz matrix and $\mathbf{u} = (u_1, \dots, u_{n+1})^T$ be a vector such that $u_1, u_{n+1} \neq 0$. If \mathbf{u} belongs to the kernel of ∂T then \mathbf{u} also belongs to the kernels of $\partial(C_t T)$, $\partial(C_b T)$, $\partial(TC_r T)$ and $\partial(TC_l T)$.*

Proof. We only prove the case $\partial(C_t T)$; the proof for the case $\partial(C_b T)$ is similar. The other two cases are covered by Theorem 2.1. Let

$$T = \begin{bmatrix} a_0 & \cdots & a_{1-n} \\ \vdots & \ddots & \vdots \\ a_{n-1} & \cdots & a_0 \end{bmatrix}$$

then, By Theorem 2.1, $C_t T$ is Toeplitz and hence

$$C_t T = \begin{bmatrix} a_{-1} & \cdots & a_{1-n} & \mu_{-1} \\ & T_{[1, n-1, 1, n]} & & \end{bmatrix},$$

where

$$\mu_{-1} = -\frac{1}{u_{n+1}}[u_n, \dots, u_1][a_{1-n}, \dots, a_0]^T.$$

It follows that

$$\partial(C_t T) = \begin{bmatrix} a_0 & \cdots & a_{1-n} & \mu_{-1} \\ & & S_1 & \end{bmatrix},$$

where S_1 is the sub-matrix of ∂T consisting of the first $n-2$ rows of ∂T . From the definition of μ_{-1} we have

$$\begin{bmatrix} a_0 & a_{-1} & \cdots & a_{1-n} & \mu_{-1} \end{bmatrix} \mathbf{u} = 0$$

and hence

$$\partial(C_t T) \mathbf{u} = 0.$$

□

Putting $H = TJ$, we have immediately

COROLLARY 2.9. *Let H be a Hankel matrix and $\mathbf{u} = (u_1, \dots, u_{n+1})^T$ be a vector such that $u_1, u_{n+1} \neq 0$. If \mathbf{u}^J belongs to the kernel of ∂H then \mathbf{u}^J belongs to the kernels of $\partial(C_t H)$ and $\partial(C_b H)$.*

The more interesting features of the extensions are now presented.

THEOREM 2.10. *Let T be an invertible Toeplitz matrix and $\mathbf{u} = (u_1, \dots, u_{n+1})^T$ be a vector such that $u_1, u_{n+1} \neq 0$. If \mathbf{u} belongs to the kernel of ∂T , then the matrix $\mathcal{T}[T : k, l; s, t]$ is Toeplitz.*

Proof. Because any $n \times n$ block in $\mathcal{T}[T : k, l; s, t]$ is in the form of $C_t^i T C_r^j$ for some integers i and j , we only need to prove that all such blocks are Toeplitz. We use the same argument in all the four directions of extension and only demonstrate this argument in one direction, say, the direction to the right. Without loss of generality we assume $i = 0$ and we prove that $T C_r^j$ is Toeplitz by induction on $j > 0$. Theorem 2.1 has already covered the case $j = 1$. Assume now all matrices $T C_r^s$, $s = 0, 1, \dots, j$ are Toeplitz. Then Lemma 2.8 guarantees that \mathbf{u} belongs to the kernel of $\partial(T C_r^j)$. Finally by Theorem 2.1 we conclude that $\partial(T C_r^{j+1})$ is Toeplitz.

In the direction to the left when $j < 0$, the same induction argument proves the case $j - 1$. The same argument works in the direction of up and down extensions.

□

COROLLARY 2.11. *Let H be an invertible Hankel matrix and $\mathbf{u} = (u_1, \dots, u_{n+1})^T$ be a vector such that $u_1, u_{n+1} \neq 0$. If \mathbf{u}^J belongs to the kernel of ∂H , then the matrix $\mathcal{H}[H : k, l; s, t]$ is Hankel.*

Proof. Define $T = HJ$ then T satisfies the conditions of Theorem 2.10 and hence this Corollary follows. □

3. Examples and applications.

3.1. Examples. For an invertible Toeplitz matrix T the kernel of ∂T is 2-dimensional and there are infinitely many choices of bases $\{\mathbf{u}, \mathbf{v}\}$ for ∂T . For any such \mathbf{u} or \mathbf{v} a companion matrix can be constructed which can be used to build extensions. Here we give three examples of obvious extensions.

Example 1. $\mathcal{T}[I : k, l; s, t]$. This extension is not necessarily Toeplitz even if the starting matrix I is Toeplitz, because \mathbf{u} is not in the kernel of ∂I unless $u_2 = \dots = u_n = 0$. However this matrix has direct application in state evolution of a dynamic system, as given later.

Example 2. $\mathcal{T}[U_+^{-1} : k, l; s, t]$. U_+^{-1} is lower triangular and we denote its elements in the first column by $s_1 \dots, s_n$. Then ∂U_+^{-1} has the form

$$\partial U_+^{-1} := \begin{bmatrix} s_2 & s_1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ s_n & \cdots & s_2 & s_1 & 0 \end{bmatrix}$$

and, obviously, \mathbf{u} is in the kernel of ∂U_+^{-1} . Therefore, $\mathcal{T}[U_+^{-1} : k, l; s, t]$ is Toeplitz. If $m = \max\{k, s\} > 0$ and $h = \max\{l, t\} > 0$, it can be shown that such an extension has the following features: (a) there is a middle band consisting of $n - 1$ diagonal lines of zeros; (b) the elements below the middle band of zeros are the elements (displayed in the same order) of the inverse of the $(n + h) \times (n + h)$ lower triangular Toeplitz matrix whose first column is $[u_1, \dots, u_{n+1}, 0, \dots, 0]^T$; (c) the elements above the middle band of zeros are the elements (displayed in the same order) of the inverse of the $m \times m$ upper triangular Toeplitz matrix whose first row is the truncation of the first m elements of $[-u_{n+1}, \dots, -u_1, 0, \dots]$. We skip the proof of this feature but demonstrate it in the case of $k = s = n$, $t = -n$ and $l = 0$. The general proof can be written down using a similar argument. We write

$$\mathcal{T}[U_+^{-1} : n, 0; n, -n] = \begin{bmatrix} S_1 & R_1 & R_2 \\ S_2 & S_1 & R_1 \end{bmatrix} = \begin{bmatrix} S_1 & 0 & 0 \\ S_2 & S_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & R_1 & R_2 \\ 0 & 0 & R_1 \end{bmatrix}.$$

It is easy to see that

$$\begin{bmatrix} S_1 & 0 \\ S_2 & S_1 \end{bmatrix} = \begin{bmatrix} U_+ & 0 \\ U_- & U_+ \end{bmatrix}^{-1}$$

because $S_1 = U_+^{-1}$ and, by Proposition 2.6, $S_2 U_+ + S_1 U_- = 0$. We now show that

$$\begin{bmatrix} R_1 & R_2 \\ 0 & R_1 \end{bmatrix} = - \begin{bmatrix} U_- & U_+ \\ 0 & U_- \end{bmatrix}^{-1}.$$

To see this we apply Proposition 2.6 to $\mathcal{T}[U_+^{-1} : n, 0; n, -n]$:

$$\begin{aligned} 0 &= \left(\begin{bmatrix} S_1 & 0 & 0 \\ S_2 & S_1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & R_1 & R_2 \\ 0 & 0 & R_1 \end{bmatrix} \right) \begin{bmatrix} U_+ & 0 \\ U_- & U_+ \\ 0 & U_- \end{bmatrix} \\ &= \begin{bmatrix} S_1 & 0 \\ S_2 & S_1 \end{bmatrix} \begin{bmatrix} U_+ & 0 \\ U_- & U_+ \end{bmatrix} + \begin{bmatrix} R_1 & R_2 \\ 0 & R_1 \end{bmatrix} \begin{bmatrix} U_- & U_+ \\ 0 & U_- \end{bmatrix}. \end{aligned}$$

The first term on the right hand side is equal to I and hence the second term on the right hand side is equal to $-I$.

Example 3. $\mathcal{T}[B_T^{-1} : k, l; s, t]$. Since both \mathbf{u} and \mathbf{v} are in the kernel of ∂B_T^{-1} there are two different version of this extension: $\mathcal{T}_u[B_T^{-1} : k, l; s, t]$ by using $C_t(\mathbf{u})$ and $C_r(\mathbf{u})$, and, $\mathcal{T}_v[B_T^{-1} : k, l; s, t]$ by using $C_t(\mathbf{v})$ and $C_r(\mathbf{v})$. Both $\mathcal{T}_u[B_T^{-1} : k, l; s, t]$ and $\mathcal{T}_v[B_T^{-1} : k, l; s, t]$ are Toeplitz and share the same central band (the diagonal band that contains only all the diagonal lines of B_T^{-1}).

3.2. Applications. For an arbitrary sequence $y = (y_k)_{k=1}^\infty$ its λ -transform (generating function) is defined to be $\hat{y}(\lambda) := \sum_{k=1}^\infty y_k \lambda^{k-1}$. Consider a linear, time-invariant, causal, discrete-time dynamic system with transfer function description

$\hat{y}(\lambda) = -(v(\lambda)/u(\lambda))\hat{x}(\lambda)$, where $x = (x_k)_{k=1}^\infty$ is the input, $y = (y_k)_{k=1}^\infty$ is the output, and the numerator and the denominator of the transfer function $-v(\lambda)/u(\lambda)$ satisfy all the assumptions stated in Section 1. We will represent this system in state space form by introducing state vectors first and then write down the rule of evolution of state vectors in terms of the initial state and the input. Known properties of the transition matrix will be derived as a special case of our results on extension of Toeplitz matrices.

When both x and y are sequences in l_1 , the analytic functions $\hat{x}(\lambda)$ and $\hat{y}(\lambda)$ are related by

$$u(\lambda)\hat{y}(\lambda) = -v(\lambda)\hat{x}(\lambda).$$

Equating like powers of λ gives

$$Uy + Vx = 0$$

where x and y are columns and

$$U := \begin{bmatrix} U_+ & & 0 \\ U_- & U_+ & \\ & U_- & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix} \quad V := \begin{bmatrix} V_+ & & 0 \\ V_- & V_+ & \\ & V_- & \ddots \\ 0 & & \ddots & \ddots \end{bmatrix}.$$

It can be shown using functional analysis arguments that such x and y have the form

$$(3.1) \quad (x, y) = (-Ub, Vb) \quad \text{for some } b \in l_1.$$

Writing out this equation in detail yields the difference equations

$$(3.2) \quad x_k = -u_{n+1}b_k - u_n b_{k+1} - \cdots - u_2 b_{k+n-1} - u_1 b_{k+n}$$

and

$$(3.3) \quad y_k = v_{n+1}b_k + v_n b_{k+1} + \cdots + v_2 b_{k+n-1} + v_1 b_{k+n}.$$

Now we can introduce naturally the n -dimensional state vector at the time k as the truncation $b_{[k; n+k-1]}$ of b and denote it by

$$\beta(k) = [b_k, b_{k+1}, \dots, b_{k+n-1}]^T.$$

Then we can put (3.2) and (3.3) in the state space form

$$(3.4) \quad \beta(k+1) = C_b \beta(k) + Bx_k, \quad y_k = D\beta(k) - \frac{v_1}{u_1}x_k$$

where the input matrix B and the output matrix D are given by

$$B = - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1/u_1 \end{bmatrix} \quad \text{and} \quad D = \frac{1}{u_1} \begin{bmatrix} u_1 v_{n+1} - v_1 u_{n+1} & \cdots & u_1 v_2 - v_1 u_2 \end{bmatrix}.$$

Note that the output matrix D is actually the first row of $B_T(\mathbf{u}, \mathbf{v})$ divided by u_1 . Using the extension given in Example 1, the general truncation $b_{[1:n+p]} = [b_1, \dots, b_{n+p}]^T$ for $p > 0$ is then given by

$$(3.5) \quad b_{[1:n+p]} = \mathcal{T}[I : 0, -p; 0, 0]\beta(0) - \frac{1}{u_1} \begin{bmatrix} O_{n \times p} \\ F_p \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix},$$

where

$$F_p = \begin{bmatrix} 1 & & & 0 \\ s_1 & 1 & & \\ \vdots & \ddots & \ddots & \\ s_{p-1} & \cdots & s_1 & 1 \end{bmatrix},$$

and s_i is the element at the last column and last row of $(C_b)^i$. It can be shown that $(1/u_1)F_p$ is the inverse of the $p \times p$ lower triangular nonsingular truncation of U , but we skip the proof here.

Now we change the basis of the state space by using the transforming matrix B_T , that is, we introduce $\beta'(k) = B_T\beta(k)$. Then, by (2.10), the state space representation of the system is transformed into another canonical form

$$(3.6) \quad \beta'(k+1) = C_l\beta'(k) + B_1x_k, \quad y_k = D_1\beta'(k) - \frac{v_1}{u_1}x_k$$

where the input matrix B_1 and the output matrix D_1 now are given by

$$B_1 = -\frac{1}{u_1} \text{Last column of } B_T \quad \text{and} \quad D_1 = \frac{1}{u_1} \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}.$$

Then the state vector at the time $q > 0$ can be expressed directly in terms of the initial state $\beta'(0)$ and the input data:

$$(3.7) \quad \beta'(q) = C_l^q\beta'(0) + \mathcal{T}[I : 0, 0; 0, 1-q]E_q \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix},$$

where E_q is the $(n+q-1) \times q$ band matrix

$$\begin{bmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_1 \end{bmatrix}.$$

Combining (3.5) and (3.7) gives the mixed case, evolving the state vectors under the basis above until a given time q , then changing the basis and evolving further to the

time $q + p$.

$$\begin{aligned}
b_{[q+1:n+q+p]} &= \mathcal{T}[I : 0, -p; 0, 0] B_T^{-1} \beta'(q) - \frac{1}{u_1} \begin{bmatrix} O_{n \times p} \\ F_p \end{bmatrix} \begin{bmatrix} x_{q+1} \\ \vdots \\ x_{q+p} \end{bmatrix} \\
&= \mathcal{T}[B_T^{-1} : 0, -p; n - q, -q] \beta'(0) + \mathcal{T}[I : 0, -p; 0, 1 - q] E_q \begin{bmatrix} x_1 \\ \vdots \\ x_q \end{bmatrix} \\
(3.8) \quad &- \frac{1}{u_1} \begin{bmatrix} O_{n \times p} \\ F_p \end{bmatrix} \begin{bmatrix} x_{q+1} \\ \vdots \\ x_{q+p} \end{bmatrix}
\end{aligned}$$

for all positive integers p and q . The expression in (3.8) is for demonstration only. It is not necessary in practice because there is no need to change basis in the middle of evolution.

All extensions involve powers of companion matrices. The formula derived in [8] can be used to calculate entries of an integer power of a companion matrix directly and hence it is possible to calculate all required entries of our extension directly without calculating all the powers of the companion matrix.

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